A Study on the Nature of Differential Equations and the Characteristics of their Solutions

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Abstract: Among many purposes of science, analyzing nature may be the most important and beautiful part. We all know that Differential Equations (DEs) are the mathematical expression of many natural phenomena. In modern science, analyzing tools like calculus, measure, sequence, series etc. have been used very frequently. Every analysis of Des has only one goal which is to get the solutions of a DE. In fact, most of these Des don't have exact solutions and many methods have been introduced to get some good solutions. Now-a-days, Functional analysis plays an important role to analyze these methods. Some methods have solid foundation and flexibility. To make use all of these methods properly, we have to understand the nature of DEs and also realize the characteristics of the solutions. Without having any idea about solutions we don't think more, this is why we have started our study of analysis to make more and more benefits of these methods. This analysis will give us a solid platform to select best methods among others and will help us to find new more accurate methods.

Index Terms: Sobolev space, σ-algebra, Hilbert norm, Sobolev inequalities, function space, ellipticity, condition

1 INTRODUCTION

1.1 Measure spaces [1]: Let X is an arbitrary set. A collection $\mathfrak{A}$ of subsets of X is called algebra (or a field) of subsets of X if it satisfies the following conditions:

(i) $X \in \mathfrak{A}$,
(ii) $A \in \mathfrak{A}$ implies that $A^c \in \mathfrak{A}$,
(iii) A, B $\in \mathfrak{A}$ implies that $A \cup B \in \mathfrak{A}$.

An algebra $\mathfrak{A}$ of subsets of a set X is called $\sigma-$algebra (or a $\sigma-$field) if it satisfies the additional condition:

(iv) $(A_n : n \in \mathbb{N}) \subset \mathfrak{A}$ Implies that $U_{n \in \mathbb{N}} A_n \in \mathfrak{A}$.

Definition-1: Let $\mathfrak{A}$ be $\sigma-$algebra of subsets of a set X. The pair (X, $\mathfrak{A}$) is called a measurable space. A subset E of X is said to be $\mathfrak{A}$-measurable if E $\in \mathfrak{A}$.

a) If μ is a measure on $\sigma-$algebra $\mathfrak{A}$ of subsets of a set X, we call the triple (X, $\mathfrak{A}$,μ) a measure space.
b) A measure μ on $\sigma-$algebra $\mathfrak{A}$ of subsets of a set X is called $\sigma-$finite measure if μ(X)< ∞. In this case, (X, $\mathfrak{A}$, μ) is called a finite measure space.
c) A measure μ on $\sigma-$algebra $\mathfrak{A}$ of subsets of a set X is called a σ-finite measure if there exists a sequence $(E_n : n \in \mathbb{N})$ in $\mathfrak{A}$ such that $U_{n \in \mathbb{N}} E_n = X$ and μ $(E_n) < \infty$ for every $n \in \mathbb{N}$. In this case, (X, $\mathfrak{A}$,μ) is called a $\sigma-$finite measure space.
d) A set D $\in \mathfrak{A}$ in an arbitrary measure space (X, $\mathfrak{A}$,μ) is called a σ-finite set if there exists a sequence $(D_n : n \in \mathbb{N})$ in $\mathfrak{A}$ such that $U_{n \in \mathbb{N}} D_n = D$ and $\mu(D_n) < \infty$ for every $n \in \mathbb{N}$.

Definition-2 [2]: Let H be a vector space. A scalar product (u, v) is a bilinear form on H ×H with values in ℝ (i.e., a map from H ×H to ℝ that is linear in both variables) such that

$$(u, v) = (v, u) \ \forall u, v \in H \ \text{(symmetry)}$$
$$(u, u) \geq 0 \quad \forall u \in H \quad \text{(positive)}$$
$$(u, u) \neq 0 \quad \forall u \neq 0 \quad \text{(definite)}$$

A bilinear form a: H×H → ℝ is said to be

i. Continuous if there is a constant C such that
$$|a(u,v)| \leq C|u||v| \qquad \forall u, v \in H;$$
ii. Coercive if there is a constant α > 0 such that
$$a(u, v-u) \geq \langle \varphi, v-u \rangle \qquad \forall v \in K.$$

Proposition [2]: Given any $f \in L^2(I)$ and α, β ∈ ℝ there exist a unique function u$\in H^2(I)$. Furthermore, u is obtained by

$$\min_{v \in H^2(I)} \left\{ \frac{1}{2} \int_I (v'^2 + v^2) - \int_I fv + \alpha v - \beta v(1) \right\}.$$

If, in addition, $f \in C(\bar{I})$, then $u \in C^2(\bar{I})$.

1.2 Maximum principle for the Dirichlet problem [2]: Assume that

$$f \in L^2(U) \text{ and } u \in H^1(U) \cap C(\bar{U})$$

Satisfy

$$\int_U \nabla u . \nabla \varphi + \int_U u\varphi = \int_U f\varphi \qquad \forall \varphi \in H_0^1(U).$$

Then for all x ∈ U,

$$\min\{\inf_{\Gamma} u, \inf_{U} f\} \ \max\{\sup_{\Gamma} u, \sup_{U} f\}$$

(Here and in the following, sup = essential sup and inf = essential inf.)

There exist a Hilbert basis $(e_n)_{n\geq 1}$ of $L^2(U)$ and a sequence and $(\lambda_n)_{n\geq 1}$ of reals with $\lambda_n > 0$ ∀n and $\lambda_n \to +\infty$ such that

$$e_n \in H_0^1(U) \cap C^\infty(U),$$
$$-\Delta e_n = \lambda_n e_n \quad \text{in U}.$$

We say that the λ_n's are the eigenvalues of -Δ (with Dirichlet boundary condition) and that the e_n's are the associated eigenfunctions.

Stampacchia [2]: Assume that a (u, v) is a continuous coercive bilinear form on H. Let K ⊂ H be a nonempty closed and convex subset. Then, given any $\varphi \in H^*$, there exists a unique element u ∈ K such that

$$a(u, v - u) \geq \langle \varphi, v-u \rangle \qquad \forall v \in K.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in k \text{ and } \tfrac{1}{2}a(u, u) - \langle \varphi, u \rangle = \min_{v \in K} \{\tfrac{1}{2}a(v, v)\langle \varphi, v \rangle\}$$

1.3 Banach fixed-point theorem the contraction mapping principle [2]: Let X be a nonempty complete metric space and let S: X→X be a strict contraction, i.e.,

$$d(Sv_1, Sv_2) \leq k d(v_1, v_2) \;\; \forall v_1, v_2 \in X \text{ with } k < 1.$$

Then S has a unique fixed point, u = Su.

Lax-Milgram [2]: Assume that a (u, v) is a continuous coercive bilinear form on H. Then, given any $\varphi \in H*$, there exists a unique element u ∈ H such that

$$a(u,v) = \langle \varphi, v \rangle \qquad \forall v \in H$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in H \text{ and}$$
$$\frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \min_{v \in H} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v \rangle \right\}$$

Differentiation of a composition [2]: Let G ∈ $C^1(\mathbb{R})$ be such that G (0) = 0 and $|G'(s)| \leq M$ ∀s ∈ ℝ for some constant M. Let $u \in W^{1,p}(U)$ with $1 \leq p < \infty$. Then

$$G \circ u \in W^{1,p}(U) \text{ and } \frac{\partial}{\partial x_i}(G \circ u) = (G' \circ u)\frac{\partial u}{\partial x_i},$$

i = 1, 2, N

1.4 Motivation for the definition of weak derivative [3]: Assume we are given u ∈ C^1(U). Then if $\phi \in C_c^\infty$(U), we see from the integration by parts formula that

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i} \phi dx \quad (i = 1\ldots\ldots n) \qquad (1)$$

There are no boundary terms, since ϕ has compact in U and thus vanishes near ∂U. More generally now, if k is a positive integer, u∈$C^k(U)$, and α = $(\alpha_1, \ldots\ldots\ldots\ldots, \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \ldots\ldots + \alpha_n = k$, then

$$\int_U u D^\alpha \phi dx = (-1)^\alpha \int_U D^\alpha u \phi dx \qquad (2)$$

This equality holds since

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots\ldots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi$$

And we can apply formula (1) $|\alpha|$ times.

We next examine formula (2), valid for u∈ $C^k(U)$, and ask whether some variant of it might be true even if u is not k times continuously differentiable. Now the left side of (2) makes sense if u is only summable: the problem is rather that if u is not C^k then the expression "$D^\alpha u$" on the right hand side of (2) has no obvious meaning. We resolve this difficulty by asking if there exists a locally summable function v for which formula (2) is valid, with v replacing $D^\alpha u$.

Let U ⊂ $\mathbb{R}^n$ be an open set and let $u \in L^1_{loc}(U)$ be such that

$$\int uf = 0 \qquad \forall f \in C_c^\infty(U)$$

Then, u = 0 a.e. on U.

Uniqueness of weak derivatives [3]: A weak α^{th} –partial derivative of u, if it is exists, is uniquely defined up to a set of measure zero.

Example [3]: Let $n = 1, U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$$

Define $v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$. Let us show that u′ = v in the weak sense. To see this, choose any $\phi \in C_c^\infty(U)$. We must demonstrate $\int_0^2 u\phi' dx = -\int_0^2 v\phi dx$.

Now, $\int_0^2 u\phi' dx$

$$= \int_0^1 x\phi' dx + \int_1^2 \phi' dx = -\int_0^1 \phi dx + \phi(1) - \phi(1) = -\int_0^2 v\phi dx.$$

Example: Let $n = 1, U = (1, 3)$,

and $u(x) = \begin{cases} x\, if & 1 < x \leq 2 \\ 3 & if\; 2 < x < 3 \end{cases}$

Define $v(x) = \begin{cases} 1 & if\; 1 < x \leq 2 \\ 0 & if\; 2 < x < 3 \end{cases}$. Let us show that $u' \neq v$ in the weak sense.

To see this, choose any $\phi \in C_c^\infty(U)$. We have to show that $\int_1^3 u\phi' dx \neq -\int_1^3 v\phi dx$.

Now, $\int_1^3 u\phi' dx = \int_1^2 x\phi' dx + 3\int_2^3 \phi' dx = -\int_1^2 \phi dx + 2\phi(2) - 3\phi(2) = -\int_1^2 \phi dx - \phi(2) = -\int_1^3 v\phi dx - \phi(1) \neq -\int_1^3 v\phi dx.$

1.5 Sobolev inequalities [3]

The crucial analytic tools here will be certain so-called "Sobolev-type inequalities", which we will prove below for smooth functions. These will then establish the estimates for arbitrary functions in the various relevant Sobolev spaces.

To clarify the presentation we will consider first only the Sobolev space $W^{1,p}$, does u automatically belong to certain other spaces? The answer will be "yes", but which other spaces depends upon whether

$$1 \leq p < n,$$
$$P = n,$$
$$n < p \leq \infty$$

Suppose that U is of class C^1. Let

$$u \in W^{1,\,p}(U) \cap C(\bar{U}) \quad \text{with } 1 \leq p < \infty.$$

Then the following properties are equivalent:

(i) u = 0 on Γ.

(ii) $u \in W_0^{1,\,p}(U)$

1.6 Holder's Inequality for $p, q \in (1, \infty)$: Given a measure space (X, 𝔄,μ). Let f and g be two extended complex-valued $\mathfrak{A}$-measurable function on X such that $|f|,|g|<\infty$ a.e. on X.

i. For any $p,q\in(1,\infty)$ such that $1/p + 1/q = 1$, we have

$$\|fg\|_1 \le \|f\|_p\|g\|_q,$$

Provided the product of the two nonnegative extended real numbers $\|f\|_p$ and $\|g\|_p$ exists.

(ii) If $0 < \|f\|_p < \|g\|_q < \infty$, then the equality in the above equation holds if and only if

$$A|f|^p = B|g|^q$$

a.e. on X for some A, B > 0.

1.7 Schwarz's Inequality: Given a measure space (X, $\mathfrak{A}$,μ). Let f and g be two extended complex-valued $\mathfrak{A}$-measurable functions on X such that $|f|,|g|<\infty$. Then we have

$$\|fg\|_1 \le \|f\|_2\|g\|_2,$$

Provided the product of the two nonnegative extended real numbers

$\|f\|_2$ and $\|g\|_2$ exists.

1.8 Inhomogeneous Neumann Condition: Consider the problem

$$\begin{cases} -u''+u=f \\ u''(0)=\alpha, u'(1)=\beta \end{cases} \quad \text{On I} = (0, 1), \qquad (3)$$

With α, β ∈ $\mathbb{R}$ are given function.

1.9 Murkowski's inequality for $p\in[1,\infty)$: Given a measure space (X, $\mathfrak{A}$,μ). Let f and g be two extended complex-valued $\mathfrak{A}$-measurable function on X such that $|f|,|g|<\infty$. Then for every $p\in[1,\infty)$, we have

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

The norm $\|\cdot\|_h = \{\langle .,. \rangle\}^{1/2}$ on an inner product space $(X,\langle .,. \rangle)$ is called a Hilbert norm. If X is complete with respect to the Hilbert norm, then X is called a Hilbert space.

2 Methodology

The maximum principle is a very useful tool, and it admits a number of formulations. We present here some simple forms. Let U be a general open subset of $\mathbb{R}^N$.

Let f: X → $\mathbb{R}$ be a real valued function defined on a set X. A real number a is called an upper bound for f if f (x) ≤ a for all x in X, i.e., if the set

$f^{-1}(a,\infty)$ = {x ∈ X: f (x) > a} is empty.

Let

$$U_f = \{a \in \mathbb{R}: f^{-1}(a,\infty) = \emptyset\}$$

be the set of upper bounds of f. Then the supremum of f is defined by

$$\sup f = \inf U_f$$

If the set of upper bounds U_f is nonempty, and sup f = + ∞ otherwise. **[4]**

Now assume in addition that (X, $\mathfrak{A}$,μ) is a measure space and, for simplicity, assume that the function f is measurable. A number a is called an essential upper bound of f if the measurable set $f^{-1}(a,\infty)$ is a set of measure zero. i.e., if f(x) ≤ a for almost all x in X.

Let

$$U_f^{ess} = \{a \in \mathbb{R}: \mu(f^{-1}(a,\infty)) = 0\}$$

Be the set of essential upper bounds. Then the essential supremum is defined similarly as

$$\text{ess sup } f = \inf U_f^{ess}$$

If $U_f^{ess} \neq \emptyset$, and ess sup f = +∞ otherwise.

Exactly in the same way one defines the essential infimum as the supremum of the essential lower bounds, that is,

$$\text{ess inf } f = \sup\{b \in \mathbb{R} : \mu((\{x : f(x) < b\}) = 0\}$$

If the set of essential lower bounds is nonempty, and as -∞ otherwise

On the real line consider the Lebesgue measure and its corresponding σ-algebra $\mathfrak{A}$.

Define a function f by the formula

$$f(x) = \begin{cases} 5, & if\ x = 1 \\ -4, & if\ x = -1 \\ 2, & otherwise \end{cases}$$

The supremum of this function (largest value) is 5, and the infimum (smallest value) is -4. However, the function takes these values only on the sets 1 and -1 respectively, which are of measure zero. Everywhere else, the function takes the value 2. Thus, the essential supremum and the essential infimum of this function are both 2.

2.1 General elliptic equations of second order

Let U ⊂ $\mathbb{R}^N$ be an open bounded set. We are given functions $a_{ij}(x) \in C^1(\overline{U})$, 1 ≤ i, j ≤ N, satisfying the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \forall \xi \in \mathbb{R}^N \text{ With } \alpha > 0 \qquad (1)$$

2.2 Maximum principle for the Dirichlet problem

Assume that

$$f \in L^2(U) \text{ and } u \in H^1(U) \cap C(\overline{U})$$

Satisfy

$$\int_U \nabla u.\nabla\varphi + \int_U u\varphi = \int_U f\varphi \qquad \forall \varphi \in H_0^1(U)$$

Then for all x ∈ U,

$$\min\{\inf_\Gamma u, \inf_U f\} \max\{\sup_\Gamma u, \sup_U f\}$$

(Here and in the following, sup = essential sup and inf = essential inf.)

We use Stampacchia's truncation method. Fix a function G $\in C^1(\mathbb{R})$ such that

(i) $|G'(s)| \le M \qquad \forall s \in \mathbb{R}$,

(ii) G *is strictly* increasing on $(0, +\infty)$,

(iii) $G(s) = 0 \qquad \forall s \le 0$.

Set

$$K = \max\{\sup_{\Gamma} u, \sup_{U} f\}$$

And assume K <∞ (otherwise there is nothing to prove).
Let v = G (u-K).
We distinguish two cases:

(a) The case $|U| < \infty$.

Then, $v \in H^1(U)$. on the other hand, v$\in H_0^1(U)$, since v $\in C(\bar{U})$ and v = 0 onΓ . Plug this v into (1) and proceed as the proof of Proposition-1.

(b) The case $|U| = \infty$.

We have then K ≥ 0 (since $f(x) \le K$ a.e. in U and $f \in L^2$ imply K ≥ 0). Fix $K' > K$. By the differentiation of the composition applied to the function $t \mapsto G(t - K')$ we see that v = G (u-K') $\in H^1$(U). Moreover, v $\in C(\bar{U})$ and v = 0 on Γ; thus v $\in H_0^1(U)$. Plugging this v into (1) we have

$$\int_U |\nabla u|^2 G'(u-K') + \int_U uG(u-K') = \int_U fG(u-K') \qquad (2)$$

On the other hand, G $(u-K') \in L^1(U)$, since

$$0 \le G(u-K') \le M|u|,$$

And on the set $[u \ge K'] = \{x \in U: u(x) \ge K'\}$ we have

$$\int_{[u\ge K']} |u| \le \int_U u^2 < \infty$$

We conclude from (2) that

$$\int_U (u-K')G(u-K') \le \int_U (f-K')G(u-K') \le 0.$$

It follows that u ≤ K' i.e. in U and thus u ≤ K a.e. in U (since $K' > K$ is arbitrary).

Suppose that the functions $a_{ij} \in L^\infty(U)$ satisfy the ellipticity condition (1) and that $a_i, a_0 \in L^\infty(U)$ with $a_0 \ge 0$ in U. Let $f \in L^\infty(U)$ and u $\in H^1 \cap C(\bar{U})$ be such that

$$\int_U \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \int_U \sum_i a_i \frac{\partial u}{\partial x_i} \varphi + \int_U a_0 u \varphi = \int_U f\varphi$$

$$\forall \varphi \in H_0^1(U) \qquad (3)$$

Then

$$[u \ge 0 \text{ on } \Gamma \text{ and } f \ge 0 \text{ in } U] \Rightarrow [u \ge 0 \text{ in } U] \qquad (4)$$

Suppose that $a_0 \equiv 0$ and that U is bounded. Then

$$[f \ge 0 \text{ in } U] \Rightarrow [u \ge \inf_{\Gamma} u \text{ in } U] \qquad (5)$$

And

$$[f = 0 \text{ in } U] \Rightarrow [\inf_{\Gamma} u \le \sup_{\Gamma} u \text{ in } U] \qquad (6)$$

We prove this result in the case $a_i \equiv 0$, 1 ≤ I ≤ N; the general case is more delicate. To establish (4) is the same as showing that

$$[u \le 0 \text{ on } \Gamma \text{ and } f \le 0 \text{ in } U] \Rightarrow [u \le 0 \text{ in } U] \qquad (7)$$

We choose φ = G (u) in (6); we thus obtain

$$\int_U \sum_{i,} a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} G'(u) \le 0,$$

And so

$$\int_U |\nabla u|^2 G'(u) \le 0.$$

Set H (t) = $\int_0^1 [G'(s)]^{1/2} ds$, so that

$$H(u) \in H_0^1(U) \text{ and } |\nabla H(u)|^2 = |\nabla u)|^2 G'(u) = 0.$$

It follows that H (u) = 0 in U and hence u ≤ 0 in U.

We now prove (5) in the following form:

$$[f \le 0 \text{ in } U] \Rightarrow [u \le \sup_{\Gamma} \text{ in } U] \qquad (8)$$

Set K = $\sup_{\Gamma} u$; then (u – K) satisfies (3), since $a_0 \equiv 0$ and (u – k)$\in H^1(U)$, since U is bounded, applying (8) we obtain u – k ≤ 0 in U, i.e., (4). Finally, (6) follows from (5) and (7).

3 Results

3.1 Homogeneous Dirichlet problem for the Laplacian

Let U $\subset \mathbb{R}^N$ be an open bounded set. We are looking for a function u: $\bar{U} \to \mathbb{R}$ satisfying

$$\begin{cases} -\nabla u + u = f \ \ in\ U \\ u = 0 \ \ on\ \Gamma = \partial U, \end{cases}$$

Where

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \text{Laplacian of u}$$

And f is a given function on U. The boundary condition u = 0 on Γ is called the (homogeneous) Dirichlet condition.
A classical solution of (4) is a function u $\in C^2(\bar{U})$ satisfying (4) (in the usual sense).

3.2 Every classical solution is a weak solution:

Indeed, u $\in H^1(U) \cap C(\bar{U})$ and u = 0 on Γ, so that u $\in H_0^1(U)$.
On the other hand, if v$\in C_c^1(U)$ we have

$$\int_U \nabla u . \nabla v + \int_U uv = \int_U fv$$

And by density this remains true for all v$\in H_0^1(U)$.

3.3 Existence and uniqueness of a weak solution

Dirichlet, Riemann, Poincare, and Hilbert:

Given any $f \in L^2(U)$, there exists a unique weak solution u $\in H_0^1(U)$ of (7). Furthermore, u is obtained by

$$\min_{v\in H_0^1(U)}\{\frac{1}{2}\int_U(|\nabla v|^2+|v|^2)-\int_U fv\}$$

This Dirichlet's is principle.

Apply Laxi-Milgram in the Hilbert space H $=H_0^1(U)$ with the bilinear form

$$a(u, v) = \int_U(\nabla u.\nabla v+uv)$$

And the linear functionalφ: v → $\int_U fv$

3.4 Regularity of the weak solution:

We say that an open set U is of class C^m, m ≥ 1 an integer, if for every x ∈ Γ there exist a neighborhood U of x in $\mathbb{R}^N$ and a bijective mapping H : Q → U such that $H\in C^m(\overline{Q})$. $H^{-1}\in C^m(\overline{U})$, H $(Q_\dagger)$ = H ∩ U, H (Q_0) = U ∩ Γ.

We say that U is of class C^∞ if it is of class C^m for all m.

The main regularity results are the following.

3.5 Regularity for the Dirichlet problem:

Let U be an open set of class C^2 with Γ bounded (or else U = $\mathbb{R}^N_\dagger$). Let $f\in C^2(U)$ and let u ∈ $H_0^1(U)$ satisfy

$$\int_U \Delta u.\Delta\varphi+\int_U u\varphi=\int_U f\varphi \qquad \forall\varphi\in H_0^1(U) \qquad (1)$$

Then, u ∈ H^2(U) and $\|u\|_{H^2}\leq C\|f\|_{L^2}$, Where C is a constant depending only on U. Furthermore, if U is of class C^{m+2} and $f\in H^m$(U), then

u ∈ H^{m+2}(U) and $\|u\|_{H^{m+2}}\leq C\|f\|_{H^m}$.

In particular, if $f\in C^\infty(\overline{U})$, then u ∈ $C^\infty(\overline{U})$.

3.6 Regularity for the Neumann problem

With the same assumptions as in equation-1 one obtains the same conclusions for the solution of the Neumann problem, i.e., for u ∈ H^1(U) such that

$$\int_U \Delta u.\Delta\varphi+\int_U u\varphi=\int_U f\varphi \quad \forall\varphi\in H^1(U) \qquad (2)$$

One would obtain the same conclusions for the solution of the Dirichlet (or Neumann) problem associated to a general second order elliptic operator, i.e., if u ∈ $H_0^1(U)$ is such that

$$\int_U\sum_{i,j}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}+\int_U\sum_i a_i\frac{\partial u}{\partial x_i}\varphi+\int_U a_0u\varphi=\int_U f\varphi$$
$$\forall\varphi\in H_0^1(U);$$

Then

$[\ f\in L^2(U),\ a_{ij}\in C^1(\overline{U})$ and $a_i\in C(\overline{U})]\Rightarrow u\in H^2(U)$,

and for m ≥ 1,

$[f\in H^m(U),\ a_{ij}\in C^{m+1}(\overline{U})$ and $a_i\in C^m(\overline{U})]\Rightarrow$ u∈ H^{m+2}(U)

3.7 Recovery of a classical solution

Assume that the weak solution u $=H_0^1(U)$ belongs to$C^2(\overline{U})$, and assume that U is of classC^1. Then u = 0 on Γ. On the other hand, we have

$$\int_U(-\Delta u+u)v=\int_U fv \qquad \forall v\in C_c^1(U)$$

And thus -Δu + u = f a.e. on U. In fact, -Δu + u = f everywhere on U, since u∈ $C^2(U)$; thus u is a classical solution.

3.8 Specification of the function space and the appropriate weak formulation

Inhomogeneous Dirichlet condition:

Let U ⊂ $\mathbb{R}^N$ be a bounded open set. We look for a function u: $\overline{U}\to\mathbb{R}$ satisfying

$$\begin{cases}-\Delta u+u=f & in\ U\\ u=g & on\ \Gamma,\end{cases} \qquad (3)$$

Where f is given on Γ. Suppose that there exists a function $\tilde{g}$ ∈ $H^1(U)$ ∩ C $(\overline{U})$ such that $\tilde{g}$ = g on Γ and consider the set

K = {v ∈ $H^1(U)$; v - $\tilde{g}$ ∈ $H_0^1(U)$}

Where K is independent of the choice of $\tilde{g}$ and depends only on g. K is a nonempty closed convex set in $H^1(U)$.

A classical solution of (3) is a function u ∈ $C^2(\overline{U})$ satisfying (3). A weak solution of (14) is a function u ∈ K satisfying

$$\int_U(\nabla u.\nabla v+uv)=\int_U fv \qquad \forall v\in H_0^1(U). \qquad (4)$$

As above, any classical solution is a weak solution.

Given any $f\in L^2(U)$, there exists a unique weak solution u ∈ K of (3).

Furthermore, u is obtained by

$$\min_{v\in K}\{\frac{1}{2}\int_U(|\nabla v|^2+|v|^2)-\int_U fv\}$$

We claim that u ∈ K is a weak solution of (3) if and only if we have

$$\int_U\nabla u.(\nabla v-\nabla u)+\int_U u(v-u)\geq\int_U f(v-u) \quad \forall v\in K \ (5)$$

Indeed, if u is a weak solution of (3) it is clear that (5) holds even with equality.

Conversely, if u ∈ K satisfies (16) we choose v = u ± w in (5) with w∈ $H_0^1(U)$, and (4) follows. We may then apply Stampacchia's theorem to conclude the proof.

3.9 General elliptic equations of second order

Let U ⊂ $\mathbb{R}^N$ be an open bounded set. We are given functions $a_{ij}(x)\in C^1(\overline{U})$, 1 ≤ i, j ≤ N, satisfying the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j\geq\alpha|\xi|^2,\ \forall\xi\in\mathbb{R}^N \text{ with } \alpha>0 \qquad (6)$$

A function $a_0\in C(\overline{U})$ is also given. We look for a function u : $\overline{U}\to\mathbb{R}$ satisfying

$$\begin{cases}\sum_{i,j=1}^N\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+a_0u=f & in\ U,\\ u=0 & on\ \Gamma.\end{cases} \qquad (7)$$

A classical solution of (7) is a function u ∈ $C^2(\overline{U})$ satisfying (7) in the usual sense. A weak solution of (7) is a function u ∈ $H_0^1(U)$ satisfying

$$\int_U(\sum_{i,j=1}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+\int_U a_0v=\int_U fv \quad \forall v\in H_0^1(U) \ (8)$$

As above, any classical solution is a weak solution. If $a_0x\geq 0$ on U then for all $f\in L^2$(U) there exists a unique weak solution u∈ H_0^1: just apply Lax-Milgram in the space H = H_0^1 with the continuous bilinear form

$$a(u, v) = \int_U a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_U a_0 uv$$

The coerciveness of a (,) comes from the ellipticity assumption, the assumption $a_0 \geq 0$, and Poincare's inequality. If the matrix (a_{ij}) is also symmetric, then the form a (,) is symmetric and u is obtained by

$$\min_{v \in H_0^1} \{\frac{1}{2}\int_U (\sum_{i,=1}^{N} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 v^2) - \int_U fv\}$$

We now consider a more general problem: find a function u: $\overline{U} \to \mathbb{R}$ satisfying

$$\begin{cases} -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + \sum_i a_i \frac{\partial u}{\partial x_i}) + a_0 u = f \quad in\ U, \\ u = 0 \quad on\ \Gamma. \end{cases} \qquad (9)$$

Where the functions $a_{ij} \in L^{\infty}$(U) satisfy the ellipticity condition and the functions (a_i), 0 ≤ i ≤ N are given in L^{∞}(U). A weak solution of (9) is a function u ∈ H_0^1 such that

$$\int_U \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_U \sum_i a_i \frac{\partial u}{\partial x_i} v + \int_U a_0 uv = \int_U fv$$

$$\forall v \in H_0^1(U) \qquad (10)$$

The associated continuous bilinear form is

$$a(u, v) = \int_U \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_U \sum_i a_i \frac{\partial u}{\partial x_i} v + \int_U a_0 uv$$

In general this form is not symmetric; in certain cases it is coercive: one may then use Lax-Milgram to obtain the existence and uniqueness of a weak solution. In the general case, even without coerciveness one still has the following.

If f = 0, then the set of solutions u ∈ H_0^1 of (10) is a finite-dimensional vector space, say of dimension d. Moreover, there exists a subspace F ⊂ $L^2(U)$ of dimension d such that

$$[(10) \text{ has a solution}] \Leftrightarrow [\int_U fv \quad \forall v \in F]$$

Fix λ > 0, large enough that the bilinear form

$$a (u, v) + \lambda \int_U uv$$

is coercive on H_0^1. For every $f \in L^2$ there exists a unique u ∈ H_0^1 satisfying

$$a (u, \varphi) + \lambda \int_U u\varphi = \int_U f\varphi \qquad \forall \varphi \in H_0^1.$$

Set u = Tf, so that T: $L^2 \to L^1$ is a compact linear operator (since U is bounded, the injection $H_0^1 \subset L^1$ is compact; Equation (10) is equivalent to

$$u = T (f + \lambda u). \qquad (11)$$

Set v = f + λu as a new unknown, and (11) becomes

$$v - \lambda Tv = f$$

The conclusion follows from Fredholm's alternative. [5]

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